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Analytical and numerical treatment of oscillatory mixed differential equations with differentiable delays and advances

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ABSTRACT

In this work, we study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, d\eta(\theta)$$

with delays $r(\theta)$ and advances $\tau(\theta)$, both differentiable. Some analytical and numerical criteria are obtained in order to guarantee that all solutions are oscillatory.

1. Introduction

The aim of this work is to study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))d\eta(\theta) \tag{1}$$

where $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized so that $\nu(-1) = \eta(-1) = 0$, and $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. Taking

$$\|\tau\| = \max\{\tau(\theta) : \theta \in [-1, 0]\},$$

the advance $\tau(\theta)$ will be assumed to satisfy

$$\tau(\theta_0) = \|\tau\| > \tau(\theta), \quad \forall \theta \neq \theta_0. \tag{2}$$

In the case of $\tau(\theta_0) > 0$, the function $\eta(\theta)$ is supposed to be atomic at θ_0, that is, such that

$$\eta(\theta_0^+) - \eta(\theta_0^-) \neq 0. \tag{3}$$

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The Eq. (1) represents the wider class of linear functional differential equations of mixed type and is considered in [1] as a basis for some mathematical applications appearing in the literature, such as in [2,3].

Letting $R = \max\{\|r\|, \|\tau\|\}$, by a solution of (1) we will mean any differentiable function $x : [-R, +\infty) \to \mathbb{R}$ which satisfies (1) for every $t \in [0, +\infty)$.

As usual, we will say that a solution $x(t)$ of (1) oscillates if it has arbitrarily large zeros. In [1] $x(t)$ is called oscillatory if there is no cone, $\mathcal{K}$, such that $x(t) \in \mathcal{K}$, eventually. Notice that for equations, both definitions coincide. When all solutions oscillate (1) will be said to be oscillatory.

By assuming that delays and advances are positive and differentiable on $[-1, 0]$, one can obtain some special criteria for having (1) oscillatory. In this paper we will analyse this case, complementing the results in [4] for the case where delays and advances are only continuous. Further theoretical results for delay equations are obtained in [5] and these can be extended in a natural way to the mixed equation.

The two main ingredients in theory of linear delay equations (see [6]) are the existence of a unique solution, for any given initial condition, and the exponential boundedness on those solutions. As is shown in [7], this is not at all the situation of a differential equation of mixed type like (1). However, under the atomicity assumption (3), one has that every oscillatory solution is exponentially bounded as $t \to \infty$ [1, Proposition 4]. This fact enables the oscillatory behaviour of (1) to be studied through the analysis of the zeros of the characteristic equation

$$\lambda = \int_{-1}^{0} \exp(-\lambda r(\theta))\mathrm{d}\nu(\theta) + \int_{-1}^{0} \exp(\lambda\tau(\theta))\mathrm{d}\eta(\theta). \tag{4}$$

In fact, if we let

$$M(\lambda) = \int_{-1}^{0} \exp(-\lambda r(\theta))\mathrm{d}\nu(\theta) + \int_{-1}^{0} \exp(\lambda\tau(\theta))\mathrm{d}\eta(\theta),$$

by [1, Corollary 5] the Eq. (1) is oscillatory if and only if $M(\lambda) \neq \lambda$, for every real λ. Therefore, if either

$$M(\lambda) > \lambda, \quad \forall \lambda \in \mathbb{R} \tag{5}$$

or

$$M(\lambda) < \lambda, \quad \forall \lambda \in \mathbb{R} \tag{6}$$

we can conclude that Eq. (1) is oscillatory.

2. Differentiable delays and advances

By an increasing (decreasing) function on an interval $[a, b]$ we will mean any nondecreasing (respectively nonincreasing) function, ϕ, such that $\phi(a) < \phi(b)$ (respectively, $\phi(a) > \phi(b)$). Assuming that $-1 \leqslant \theta_1 \leq 0$, let $D^+(\theta_1)$ be the family of all positive differentiable functions, which are increasing on $[-1, \theta_1]$ and decreasing on $[\theta_1, 0]$. If $\theta_1 = 0$, we obtain the set, D_i^+ of all positive increasing differentiable functions on the interval $[-1, 0]$. In the case where $\theta_1 = -1$, we obtain the class D_d^+ of all decreasing positive differentiable functions on $[-1, 0]$.

For $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$ with θ_0 as in (2), we define the value

$$S_1 = \mathrm{e}^{-1}\left(\int_{-1}^{0} \nu(\theta)\mathrm{d}\ln r(\theta) + \int_{-1}^{0} \eta(\theta)\mathrm{d}\ln\tau(\theta)\right).$$

Through (5) we obtain the following theorems.

Theorem 2.1. *For $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$, let*

$$\nu(\theta) \leqslant 0 \quad \text{for } \theta \in [-1, \theta_1[, \qquad \nu(\theta) \geqslant 0 \quad \text{for } \theta \in [\theta_1, 0] \tag{7}$$

$$\eta(\theta) \leqslant 0 \quad \text{for } \theta \in [-1, \theta_0[, \qquad \eta(\theta) \geqslant 0 \quad \text{for } \theta \in [\theta_0, 0], \tag{8}$$

such that $\eta(0) > 0$. If

$$1 + \ln(\tau(0)\eta(0)) + \tau(0)S_1 > 0 \tag{9}$$

then the Eq. (1) is oscillatory.

Proof. For $\lambda = 0$, we have $M(0) = \nu(0) + \eta(0) > 0$. Let $\lambda \neq 0$. Using integration by parts we obtain

$$M(\lambda) = \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + \lambda\int_{-1}^{0} \exp(-\lambda r(\theta))\nu(\theta)\mathrm{d}r(\theta) - \lambda\int_{-1}^{0} \exp(\lambda\tau(\theta))\eta(\theta)\mathrm{d}\tau(\theta). \tag{10}$$

Since $\nu(\theta)r'(\theta) \leqslant 0$ and $\eta(\theta)\tau'(\theta) \leqslant 0$ for $\theta \in [-1, 0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$M(\lambda) \geqslant \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + S_1.$$

Therefore

$$\begin{aligned} M(\lambda)-\lambda &\geqslant \exp(-\lambda r(0))\nu(0)+\exp(\lambda\tau(0))\eta(0)-\lambda+S_1 \\ &\geqslant \exp(\lambda\tau(0))\eta(0)-\lambda+S_1. \end{aligned} \tag{11}$$

As $\eta(0)>0$, the function $f(\lambda)=\exp(\lambda\tau(0))\eta(0)-\lambda$ attains an absolute minimum at

$$\lambda_0=-\frac{\ln(\tau(0)\eta(0))}{\tau(0)}$$

and consequently

$$M(\lambda)-\lambda\geqslant\frac{1}{\tau(0)}+\frac{1}{\tau(0)}\ln(\tau(0)\eta(0))+S_1>0.$$

Thus (5) is satisfied, which completes the proof. □

Example 2.1. Consider the Eq. (1) for $\nu(\theta)=(3\theta+1)(\theta+1)$, $\eta(\theta)=(\theta+1)(2\theta+1)$,

$$r(\theta)=-\frac{3}{2}\theta^2-\theta+1$$

and

$$\tau(\theta)=-\theta^2-\theta+2.$$

As

$$\begin{aligned} S_1 &= e^{-1}\int_{-1}^{0}(3\theta+1)(\theta+1)\frac{-3\theta+1}{-\frac{3}{2}\theta^2-\theta+1}d\theta+e^{-1}\int_{-1}^{0}(\theta+1)(2\theta+1)\frac{-2\theta-1}{-\theta^2-\theta+2}d\theta \\ &\approx -0.1421, \end{aligned}$$

$$1+\ln(\tau(0)\eta(0))+\tau(0)S_1=1+\ln 2+2S_1\approx 1.4089,$$

the corresponding Eq. (1) is oscillatory.

Example 2.2. Consider the Eq. (1) with

$$\nu(\theta)=\begin{cases}-\theta-1, & \text{if } \theta\in[-1,0[\\ 1, & \text{if } \theta=0\end{cases}$$

$$\eta(\theta)=\theta+1,\qquad r(\theta)=\theta+2\quad\text{and}\quad\tau(\theta)=-\theta+1.$$

The corresponding equation is oscillatory since

$$S_1=e^{-1}\int_{-1}^{0}\frac{-\theta-1}{\theta+2}d\theta+e^{-1}\int_{-1}^{0}\frac{-(\theta+1)}{-\theta+1}d\theta\approx-0.25499$$

and

$$1+\ln(\tau(0)\eta(0))+\tau(0)S_1=1+\ln 1+S_1\approx 0.74501.$$

Now let

$$S_2=\int_{-1}^{0}\nu(\theta)dr(\theta)-\int_{-1}^{0}\eta(\theta)d\tau(\theta).$$

Theorem 2.2. *Let $r\in D^+(\theta_1)$, $\tau\in D^+(\theta_0)$. If (7)–(8) hold such that $\nu(0)+\eta(0)>0$ and*

$$1-e\tau(0)\eta(0)<S_2<1+er(0)\nu(0) \tag{12}$$

then Eq. (1) is oscillatory.

Proof. The case where $\lambda=0$, follows as in the proof of Theorem 2.1.
For $\lambda\neq 0$, by (10) we have

$$\frac{M(\lambda)}{\lambda}=\frac{\exp(-\lambda r(0))}{\lambda}\nu(0)+\frac{\exp(\lambda\tau(0))}{\lambda}\eta(0)+\int_{-1}^{0}\exp(-\lambda r(\theta))\nu(\theta)dr(\theta)-\int_{-1}^{0}\exp(\lambda\tau(\theta))\eta(\theta)d\tau(\theta). \tag{13}$$

Let $\lambda > 0$. Since $\exp(-u) < 1$, $\exp u > 1$, $\frac{\exp(-u)}{u} > 0$ and $\frac{\exp u}{u} \geq e$, for $u > 0$, we obtain

$$\frac{M(\lambda)}{\lambda} > e\tau(0)\eta(0) + S_2 > 1$$

and so $M(\lambda) > \lambda$.

For $\lambda < 0$, the same arguments imply that

$$\frac{M(\lambda)}{\lambda} < -er(0)\nu(0) + S_2 < 1$$

and $M(\lambda) > \lambda$. Hence (5) is again satisfied and (1) is oscillatory. □

Example 2.3. Consider the Eq. (1) with

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \qquad \eta(\theta) = (10\theta + 9)(\theta + 1),$$
$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5$$

and

$$\tau(\theta) = -5\theta^2 - 9\theta + 1.$$

We have

$$S_2 = -\int_{-1}^{0}(5\theta + 4)^2(\theta + 1)\mathrm{d}\theta + \int_{-1}^{0}(10\theta + 9)^2(\theta + 1)\mathrm{d}\theta \approx 15.417$$

and

$$-23.465 \approx 1 - 9e = 1 - e\tau(0)\eta(0) < S_2 < 1 + er(0)\nu(0) = 1 + 20e \approx 55.366.$$

So, the corresponding equation is oscillatory. Notice that in this case as

$$S_1 = \mathrm{e}^{-1}\left(\int_{-1}^{0}\frac{-(5\theta + 4)^2(\theta + 1)}{-\frac{5}{2}\theta^2 - 4\theta + 5}\mathrm{d}\theta + \int_{-1}^{0}\frac{-(10\theta + 9)^2(\theta + 1)}{-5\theta^2 - 9\theta + 1}\mathrm{d}\theta\right)$$
$$\approx -3.6737$$

and

$$1 + \ln(\tau(0)\eta(0)) + \tau(0)S_1 = 1 + \ln 9 + S_1 = -0.47648 < 0$$

we cannot apply Theorem 2.1.

With respect to condition (6) we obtain the following theorem.

Theorem 2.3. *Let $r \in D^+(\theta_1)$, $\tau \in D^+(\theta_0)$ and*

$$\nu(\theta) \geqslant 0 \quad \text{for } \theta \in [-1, \theta_1[, \qquad \nu(\theta) \leqslant 0 \quad \text{for } \theta \in [\theta_1, 0], \tag{14}$$
$$\eta(\theta) \geqslant 0 \quad \text{for } \theta \in [-1, \theta_0[, \qquad \eta(\theta) \leqslant 0 \quad \text{for } \theta \in [\theta_0, 0] \tag{15}$$

such that $\nu(0) < 0$. If

$$1 + \ln(r(0)|\nu(0)|) - r(0)S_1 > 0 \tag{16}$$

then the Eq. (1) is oscillatory.

Proof. For $\lambda = 0$, we have $M(0) = \nu(0) + \eta(0) < 0 = \lambda$.

Let $\lambda \neq 0$. Applying (10) and taking into account that now $\nu(\theta)r'(\theta) \geqslant 0$ and $\eta(\theta)\tau'(\theta) \geqslant 0$ for $\theta \in [-1, 0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$M(\lambda) \leqslant \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + S_1.$$

Notice that, in this case, $M(\lambda) \to -\infty$, as $\lambda \to \pm\infty$.

Therefore

$$M(\lambda) - \lambda \leq \exp(-\lambda r(0))\nu(0) - \lambda + S_1. \tag{17}$$

The function $g(\lambda) = \exp(-\lambda r(0))\nu(0) - \lambda$ has a maximum at

$$\lambda_0 = \frac{\ln(r(0)|\nu(0)|)}{r(0)}$$

and consequently by (16)

$$M(\lambda) - \lambda \leqslant -\frac{1}{r(0)} - \frac{1}{r(0)} \ln(r(0)|\nu(0)|) + S_1 < 0,$$

for every $\lambda \in \mathbb{R}$.

Thus (6) is satisfied and (1) is oscillatory. □

Remark 2.1. Notice that conditions (7) and (8) of Theorem 2.1, by (11), imply that $M(\lambda) - \lambda \to +\infty$, as $\lambda \to \pm\infty$. Analogously to (14) and (15) of Theorem 2.3, by (17), one has $M(\lambda) - \lambda \to -\infty$, as $\lambda \to \pm\infty$. This means that in such situations, the real roots of the characteristic equation (4) are bounded.

Example 2.4. Consider the Eq. (1) with

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \qquad \eta(\theta) = -8\theta - 8,$$
$$r(\theta) = -2\theta^2 - 3\theta + 1,$$

and

$$\tau(\theta) = -\theta + 1.$$

Notice that

$$S_1 = \mathrm{e}^{-1} \int_{-1}^{0} \frac{(-\theta - 1)(4\theta + 3)(-4\theta - 3)}{-2\theta^2 - 3\theta + 1} \mathrm{d}\theta + \mathrm{e}^{-1} \int_{-1}^{0} \frac{8\theta + 8}{-\theta + 1} \mathrm{d}\theta \approx 1.6372$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0)S_1 = 1 + \ln 3 - S_1 \approx 0.4614.$$

By Theorem 2.3, the corresponding Eq. (1) is oscillatory.

Example 2.5. Consider

$$\nu(\theta) = \begin{cases} \theta + 1, & \text{if } \theta \in [-1, 0[\\ -1, & \text{if } \theta = 0, \end{cases}$$
$$\eta(\theta) = -\theta - 1, \qquad r(\theta) = -\theta^2 + 2 \quad \text{and} \quad \tau(\theta) = -\theta + 3.$$

The Eq. (1) is oscillatory since

$$S_1 = \mathrm{e}^{-1} \left(\int_{-1}^{0} \frac{-2\theta(\theta + 1)}{-\theta^2 + 2} \mathrm{d}\theta + \int_{-1}^{0} \frac{\theta + 1}{-\theta + 3} \mathrm{d}\theta \right) \approx 0.1291,$$

and

$$1 + \ln(r(0)|\nu(0)|) - r(0)S_1 = 1 + \ln 2 - 2S_1 \approx 1.4349.$$

Theorem 2.4. *Let $r \in D^+(\theta_1)$, $\tau \in D^+(\theta_0)$ and assume that (14)–(15) are satisfied such that $\nu(0) + \eta(0) < 0$. If*

$$1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) \tag{18}$$

then the Eq. (1) is oscillatory.

Proof. When $\lambda = 0$, as before one has $M(0) = \nu(0) + \eta(0) < 0$.

Let $\lambda > 0$. Using (13) and the arguments as in Theorem 2.2, we obtain

$$\frac{M(\lambda)}{\lambda} < e\tau(0)\eta(0) + S_2,$$

and by (18) follows that $M(\lambda) < \lambda$.

For $\lambda < 0$, the same arguments as before enable us to conclude that

$$\frac{M(\lambda)}{\lambda} > er(0)|\nu(0)| + S_2 > 1.$$

So, by (18) one has also $M(\lambda) < \lambda$, which achieves the proof. □

For the case where $\theta_0 = \theta_1 = -1$, the delays and advances are in $D^+_{d.}$. When $\theta_0 = \theta_1 = 0$, the delays and advances are in D^+_i. The following example illustrates this situation for Theorem 2.4.

Example 2.6. Let the Eq. (1)

$$\nu(\theta) = -(5\theta + 1)(\theta + 1), \qquad \eta(\theta) = -(6\theta + 1)(\theta + 1),$$
$$r(\theta) = -10\theta^2 - 4\theta + 10,$$

and

$$\tau(\theta) = -3\theta^2 - \theta + 1.$$

We have

$$\begin{aligned} S_2 &= \int_{-1}^{0} (5\theta + 1)(\theta + 1)(20\theta + 4)\mathrm{d}\theta - \int_{-1}^{0} (6\theta + 1)^2(\theta + 1)\mathrm{d}\theta \\ &\approx 2.1667 \end{aligned}$$
$$-26.138 \approx 1 - 10e = 1 + er(0)\nu(0) < S_2 < 1 - e\tau(0)\eta(0) = 1 + e \approx 3.7183,$$

so, the corresponding equation is oscillatory.

3. Numerical experiments

In this section, we show how numerical approximations can be used to derive information about oscillation or non-oscillation of solutions to a mixed-type equation. To begin, we give an overview of the approach, which builds on that adopted in [8]. We give more details later.

The general approach is to derive a discrete system that approximates the underlying mixed-type equation and to analyse the behaviour of solutions of the discrete scheme. The approach we have adopted here is to use a very simple discretization, based on an Euler rule to approximate the derivative on the left hand side of the equation, and a trapezoidal rule to approximate the integrals on the right hand side. In principle, one could use a more complicated approach, but the results we obtain here are very good and the method is already effective in our view.

As a general principle, we shall use a fixed step length $h > 0$ and the resulting system of discrete equation will take the form of difference equations or a recurrence relation. This can be analysed using its characteristic equation and (for no oscillatory solutions) we are looking for the case when there are no nonnegative real characteristic roots.

The root counting method we have adopted (see [8] for further discussion) is based on an application of the argument principle and Rouché's Theorem to count zeros of a polynomial function inside a closed path. We choose a rectangular path with vertices at $0 \pm \frac{1}{M}i, M \pm \frac{1}{M}i$ for large positive values of $M \in \mathbb{R}$ and count the zeros inside the rectangle as $M \to \infty$. As we saw in [8], one can show that the characteristic polynomial of the discrete problem has zeros close to the positive real axis only if the characteristic equation of the underlying continuous problem has characteristic values close to the real axis. Further details of the analytical results will be found in [8] (see also [9–12]).

For the detail, consider the numerical scheme for the Eq. (1)

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta)$$

where, $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized in manner that $\nu(-1) = \eta(-1) = 0$, $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. We shall use the backward Euler method to approximate the time derivative and use the trapezoidal method to approximate the integral. Then we obtain the corresponding discrete characteristic polynomial. Further we use Rouché's Theorem to find the numbers of real roots of the discrete characteristic polynomial. We observe that the Eq. (1) is oscillatory if and only if the characteristic polynomial has no real roots, which is consistent with the theoretic results.

Below we will describe how to find the discrete characteristic polynomial of (1). In all the numerical examples, we assume that $r(\theta)$, $\nu(\theta)$, $\tau(\theta)$ and $\eta(\theta)$ are quadratic polynomials.

Let us consider how to discretize the integral $\int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta)$. A similar idea can be applied to the integral $\int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta)$.

We first need to find the critical point θ_r of $r(\theta)$ on $[-1, 0]$, i.e., $r'(\theta_r) = 0$. Assume that $r(\theta)$ attains its maximum value at θ_r, i.e., $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$. We also assume that

$$r(-1) = r_{-1} > 0, \qquad r(0) = r_0 > 0.$$

Obviously, in this case $r(\theta_r) = r_c \geq \max\{r_{-1}, r_0\}$.

We write the integral in two parts:

$$\int_{-1}^{0} x(t-r(\theta))\,\mathrm{d}\nu(\theta) = \int_{-1}^{\theta_r} x(t-r(\theta))\,\mathrm{d}\nu(\theta) + \int_{\theta_r}^{0} x(t-r(\theta))\,\mathrm{d}\nu(\theta).$$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots$ be time points and let $h = t_{j+1} - t_j$ be the time step.

The idea of the discretization of the integral $\int_{-1}^{\theta_r} x(t+r(\theta))\,\mathrm{d}\nu(\theta)$ is to find two nonnegative integers $N_1,\ N_2,\ N_1 > N_2$ such that

$$-1 = \theta_{-N_1} < \theta_{-N_1+1} < \cdots < \theta_{-N_2} = \theta_r,$$

is a partition of $[-1, \theta_r]$ and

$$r(\theta_{-N_1}) = r(-1) = r_{-1} = N_1 h, \tag{19}$$

$$r(\theta_j) = N_1 h + m_r(N_1 + j)h, \quad j = -N_1+1, -N_1+2, \ldots, -N_1 + (N_1 - N_2 - 1), \tag{20}$$

$$r(\theta_{-N_2}) = r(\theta_r) = r_c = N_1 h + m_r(N_1 - N_2)h. \tag{21}$$

Here m_r is some positive integer which guarantees that $N_2 \geq 0$. Such N_1 and N_2 can be obtained by (19) and (21),

$$N_1 = \frac{r_{-1}}{h}, \qquad N_2 = \frac{(m_r+1)N_1 - \frac{r_c}{h}}{m_r}. \tag{22}$$

Note that $\theta_j,\ j = -N_1, -N_1+1, \ldots, -N_1 + (N_1 - N_2)$ can be obtained by solving (19)–(21) for the given $r(\theta)$.

The idea of the discretization of the integral $\int_{\theta_r}^{0} x(t+r(\theta))\,\mathrm{d}\nu(\theta)$ is to find two nonnegative integers $N_3,\ N_4$ such that

$$\theta_r = \theta_{-N_3} < \theta_{-N_3+1} < \cdots < \theta_{-1} < \theta_0 = 0,$$

is a partition of $[\theta_r, 0]$ and

$$r(\theta_0) = r(0) = r_0 = N_4 h, \tag{23}$$

$$r(\theta_l) = (N_3 h + N_4 h) - (N_3 + l)h, \quad l = -N_3+1, -N_3+2, \ldots, -1, \tag{24}$$

$$r(\theta_{-N_3}) = r(\theta_r) = r_c = N_3 h + N_4 h. \tag{25}$$

Such N_3 and N_4 can be obtained by (23) and (25),

$$N_4 = \frac{r_0}{h}, \qquad N_3 = \frac{r_c}{h} - N_4. \tag{26}$$

Note that $\theta_l,\ l = -N_3, -N_3+1, \ldots, -1, 0$ can be obtained by solving (23)–(25) for the given $r(\theta)$.

Now we can discretize the integral $\int_{-1}^{0} x(t+r(\theta))\,\mathrm{d}\nu(\theta)$ at $t = t_n$. We have

$$\begin{aligned}
\int_{-1}^{0} x(t_n - r(\theta))\,\mathrm{d}\nu(\theta) &= \int_{-1}^{\theta_r} x(t_n - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{\theta_r}^{0} x(t_n - r(\theta))\,\mathrm{d}\nu(\theta) \\
&\approx \sum_{j=-N_1}^{-N_2-1} x(t_n - r(\theta_j))\Big(\nu(\theta_{j+1}) - \nu(\theta_j)\Big) + \sum_{l=-N_3}^{-1} x(t_n - r(\theta_l))\Big(\nu(\theta_{l+1}) - \nu(\theta_l)\Big) \\
&= \sum_{j=-N_1}^{-N_2-1} x(nh - [N_1 h + m_r(N_1 + j)h])\Big(\nu(\theta_{j+1}) - \nu(\theta_j)\Big) \\
&\quad + \sum_{l=-N_3}^{-1} x(nh - [N_3 h + N_4 h - (N_3 + l)h])\Big(\nu(\theta_{l+1}) - \nu(\theta_l)\Big).
\end{aligned}$$

Similarly we can discretize the integral $\int_{-1}^{0} x(t+\tau(\theta))\,\mathrm{d}\eta(\theta)$. Now let us summarize the steps to find the characteristic polynomial of (1).

Step 1. Find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Without loss of the generality, we assume that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} > 0,\ r(0) = r_0 > 0$.

Step 2. Find the nonnegative integers $N_1, N_2,\ N_1 > N_2$ by

$$N_1 = \frac{r_{-1}}{h}, \qquad N_2 = \frac{(m_r+1)N_1 - \frac{r_c}{h}}{m_r},$$

where $r_c = r(\theta_r)$ and m_r is some positive integer such that $N_2 \geq 0$.

Find the nonnegative integers N_3 and N_4 by

$$N_4 = \frac{r_0}{h}, \qquad N_3 = \frac{r_c}{h} - N_4.$$

Step 3. Find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. Without loss of the generality, we assume that $\tau(\theta)$ is increasing on $[-1, \theta_\tau]$ and decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} > 0,\ \tau(0) = \tau_0 > 0$.

Step 4. Find the nonnegative integers $M_1, M_2,\ M_1 > M_2$ by

$$M_1 = \frac{\tau_{-1}}{h}, \qquad M_2 = \frac{(m_\tau + 1)M_1 - \frac{\tau_c}{h}}{m_\tau},$$

where $\tau_c = \tau(\theta_\tau)$ and m_τ is some positive integer such that $M_2 \geq 0$.

Find the nonnegative integers M_3 and M_4 by

$$M_4 = \frac{\tau_0}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4.$$

Step 5. Approximating the time derivative in (1) by the backward Euler method and approximating the integral in (1) by the Trapezoidal method, we obtain, at time t_n,

$$\begin{aligned}\frac{x(t_{n+1}) - x(t_n)}{h} \approx{}& \sum_{j=-N_1}^{-N_2-1} x\big(nh - [N_1 h + m_r(N_1 + j)h]\big)\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big)\\ &+ \sum_{l=-N_3}^{-1} x\big(nh - [N_3 h + N_4 h - (N_3 + l)h]\big)\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)\\ &+ \sum_{j=-M_1}^{-M_2-1} x\big(nh + [M_1 h + m_\tau(M_1 + j)h]\big)\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big)\\ &+ \sum_{l=-M_3}^{-1} x\big(nh + [M_3 h + M_4 h - (M_3 + l)h]\big)\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big).\end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = N_1 h + m_r(N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = (N_3 h + N_4 h) - (N_3 + l)h, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$r(\theta'_j) = M_1 h + m_\tau(M_1 + j)h, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$r(\theta'_l) = (M_3 h + M_4 h) - (M_3 + l)h, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Denote $x^n \approx x(t_n),\ n = 0, 1, 2, \ldots$. We have

$$\begin{aligned}\frac{x^{n+1} - x^n}{h} ={}& \sum_{j=-N_1}^{-N_2-1} x^{n-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} x^{n-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)\\ &+ \sum_{j=-M_1}^{-M_2-1} x^{n+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) + \sum_{l=-M_3}^{-1} x^{n+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big).\end{aligned}$$

Denote $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$. Choosing $n = N$ and replacing x by z, we get the following discrete characteristic equation of (1)

$$\begin{aligned}P(z) ={}& -z^{N+1} + z^N + h\Bigg[\sum_{j=-N_1}^{-N_2-1} z^{n-[N_1+m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} z^{n-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)\\ &+ \sum_{j=-M_1}^{-M_2-1} z^{n+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) + \sum_{l=-M_3}^{-1} z^{n+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Bigg].\end{aligned}$$

Step 6. Apply Rouché's Theorem to determine the existence of the positive real roots of the characteristic polynomial $P(z)$.

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{P'(z)}{P(z)}\,\mathrm{d}z = \text{Number of zeros of } P(z) \text{ inside the closed curve } C.$$

In our numerical simulation, we chose the curve C as the boundary of a rectangle with vertices $A\left(0, \frac{1}{M}\right)$, $B\left(0, -\frac{1}{M}\right)$, $C\left(M, -\frac{1}{M}\right)$ and $D\left(M, \frac{1}{M}\right)$ for some sufficiently large M.

Remark 3.1. We can use a similar idea to work on the case where $r(-1) = r_{-1} < 0$ and $r(0) = r_0 < 0$, or $r(-1) \cdot r(0) < 0$.

Remark 3.2. We can also use a similar idea to work on the case where $r(\theta)$ (or $\tau(\theta)$) is decreasing on $[-1, \theta_c]$ (or $[-1, \theta_\tau]$) and increasing on $[\theta_c, 0]$ (or $[\theta_\tau, 0]$).

Below we will consider how to construct the discrete characteristic polynomials for some examples.

Example 3.1. Consider the Eq. (1) with the conditions of Example 2.1

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, \mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, \mathrm{d}\eta(\theta). \tag{27}$$

Here

$$\nu(\theta) = (3\theta + 1)(\theta + 1), \qquad \eta(\theta) = (\theta + 1)(2\theta + 1),$$

and

$$r(\theta) = -\frac{3}{2}\theta^2 - \theta + 1, \qquad \tau(\theta) = -\theta^2 - \theta + 2.$$

Let us find the discrete characteristic polynomial of (27). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -3\theta - 1 = 0$. we get $\theta_r = -\frac{1}{3}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} = \frac{1}{2} > 0,\ r(0) = r_0 = 1 > 0$ and $r(\theta_r) = r_c = \frac{7}{6}$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{2h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = \frac{1}{6h},$$

where we choose $m_r = 2$ which guarantees that $N_2 > 0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4 = \frac{r_0}{h} = \frac{1}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{1}{6h}.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. Let $\tau'(\theta) = -2\theta - 1 = 0$. we get $\theta_\tau = -\frac{1}{2}$. Further it is easy to find that $\tau(\theta)$ is increasing on $[-1, \theta_\tau]$ and decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 2 > 0,\ \tau(0) = \tau_0 = 2 > 0$ and $\tau(\theta_\tau) = \tau_c = 2.25$.

The nonnegative integers $M_1, M_2,\ M_1 > M_2$ can be determined by

$$M_1 = \frac{\tau_{-1}}{h} = \frac{2}{h}, \qquad M_2 = \frac{(m_\tau + 1)M_1 - \frac{\tau_c}{h}}{m_\tau} = \frac{7}{4h},$$

where we choose $m_\tau = 1$ which guarantees that $M_2 \geq 0$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{2}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{4h}.$$

Finally we denote $N = \max\{N_1 + 2(N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (27)

$$P(z) = -z^{N+1} + z^N + h\Bigg[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)$$
$$+ \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Bigg].$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -\frac{3}{2}\theta_j^2 - \theta_j + 1 = (3N_1 + 2j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -\frac{3}{2}\theta_l^2 - \theta_l + 1 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0$$

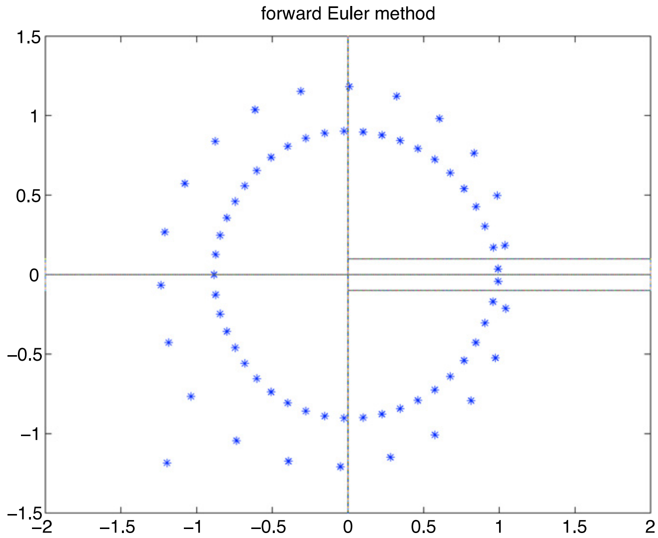

Fig. 1. Characteristic plot for $h = 0.05$, $M = 10$.

Table 1
Cf. Fig. 1: Number of zeros of the polynomial by Rouché's Theorem.

Step length h	Length of rectangle M	Number of zeros N_P
0.05	2	12
0.05	4	6
0.05	10	2
0.05	20	2
0.05	Large	0

which implies that

$$\theta_j = \frac{1+\sqrt{1+6\big(1-(3N_1+2j)h\big)}}{2\times(-3/2)}, \quad j = -N_1, -N_1+1, \dots, -N_2,$$

and

$$\theta_l = \frac{1-\sqrt{1+6\big(1-N_4h+lh\big)}}{2\times(-3/2)}, \quad l = -N_3, -N_3+1, \dots, -1, 0.$$

Similarly, θ_j' and θ_l' are determined by

$$\theta_j' = \frac{1+\sqrt{1+4\big(2-(2M_1+j)h\big)}}{2\times(-1)}, \quad j = -M_1, -M_1+1, \dots, -M_2,$$

and

$$\theta_l' = \frac{1-\sqrt{1+4\big(2-M_4h+lh\big)}}{2\times(-1)}, \quad l = -M_3, -M_3+1, \dots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots and so this satisfies the conditions for discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillatory property of the Eq. (27). See Fig. 1 and Table 1.

Example 3.2. Consider the Eq. (1) for the Example 2.2

$$x'(t) = \int_{-1}^{0} x(t-r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t+\tau(\theta))\,\mathrm{d}\eta(\theta). \tag{28}$$

Here

$$\nu(\theta) = \begin{cases} -\theta-1, & -1 \le \theta < 0, \\ 1, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = \theta + 1, \qquad r(\theta) = \theta + 2, \qquad \tau(\theta) = -\theta + 1.$$

We now find the discrete characteristic polynomial of (28). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. We get $\theta_r = 0$. Further it is easy to see that $r(\theta)$ is increasing on $[-1, \theta_r]$ and $r(-1) = r_{-1} = 1 > 0$, $r(0) = r_0 = 2 > 0$ and $r(\theta_r) = r_c = 2$.

The nonnegative integers N_1, N_2, $N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 0,$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -1$. Further it is easy to see that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 2 > 0$, $\tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3, M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

Finally we denote $N = N_1 + (N_1 - N_2 - 1)$. Then we obtain the following discrete characteristic equation of (28)

$$P(z) = -z^{N+1} + z^N + h\left[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\right]. \tag{29}$$

Here θ_j are determined by

$$r(\theta_j) = \theta_j + 2 = N_1 h + (N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

which implies that

$$\theta_j = N_1 h + (N_1 + j)h - 2, \quad j = -N_1, -N_1 + 1, \ldots, -N_2.$$

Similarly, θ'_l are determined by

$$\tau(\theta'_l) = -\theta'_l + 1 = (M_3 h + M_4 h) - (M_3 + l)h, \quad j = -M_3, -M_3 + 1, \ldots, -1, 0$$

which implies that

$$\theta'_l = -(M_3 h + M_4 h) + (M_3 + l)h + 1, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Remark 3.3. Note that $\nu(\theta)$ has a jump at $\theta = 0$, therefore we have, in (29),

$$\begin{aligned}\nu(\theta_{-N_2}) - \nu(\theta_{-N_2-1}) &= \nu(0) - \nu(N_1 h + N_1 h - N_2 h - h)\\ &= \nu(0) - \nu(2 - h) = 1 - \big(-(2 - h) - 1\big)\\ &= 4 - h.\end{aligned}$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots and therefore satisfies the conditions for the discrete equation to be oscillatory. Hence the numerical results are consistent with the theoretical results about the oscillation property for the Eq. (28). See Fig. 2.

Example 3.3. Consider the Eq. (1) for the Example 2.3

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\, \mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\, \mathrm{d}\eta(\theta). \tag{30}$$

Here

$$\nu(\theta) = (5\theta + 4)(\theta + 1), \qquad \eta(\theta) = (10\theta + 9)(\theta + 1),$$

and

$$r(\theta) = -\frac{5}{2}\theta^2 - 4\theta + 5, \qquad \tau(\theta) = -5\theta^2 - 9\theta + 1.$$

We now find the discrete characteristic polynomial of (30). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -5\theta - 4 = 0$. We get $\theta_r = -\frac{4}{5}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} = 6.5 > 0$, $r(0) = r_0 = 5 > 0$ and $r(\theta_r) = r_c = 6.6$.

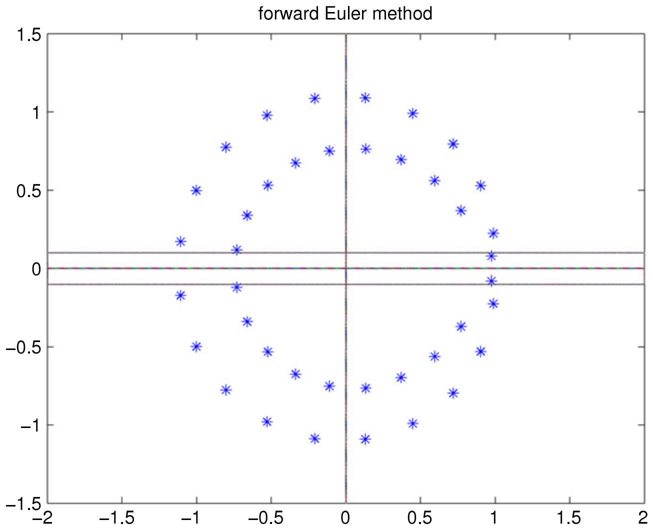

Fig. 2. Characteristic plot for $h = 0.01,\ M = 8$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{6.5}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{6.6}{h},$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4 = \frac{r_0}{h} = \frac{5}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{6.6}{h} - N_4.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. Let $\tau'(\theta) = -10\theta - 9 = 0$. we get $\theta_\tau = -\frac{9}{10}$. Further it is easy to find that $\tau(\theta)$ is increasing on $[-1, \theta_\tau]$ and decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 5 > 0,\ \tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = 5.05$.

The nonnegative integers $M_1, M_2,\ M_1 > M_2$ can be determined by

$$M_1 = \frac{\tau_{-1}}{h} = \frac{5}{h}, \qquad M_2 = \frac{(m_\tau + 1)M_1 - \frac{\tau_c}{h}}{m_\tau} = 2M_1 - \frac{5.05}{h},$$

where we choose $m_\tau = 1$ which guarantees that $M_2 \geq 0$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{2}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{4h}.$$

Finally we denote $N = \max\{N_1 + 2(N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (30)

$$P(z) = -z^{N+1} + z^N + h\Bigg[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)$$
$$+ \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Bigg].$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -\frac{5}{2}\theta_j^2 - 4\theta_j + 5 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -\frac{5}{2}\theta_j^2 - 4\theta_j + 5 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0$$

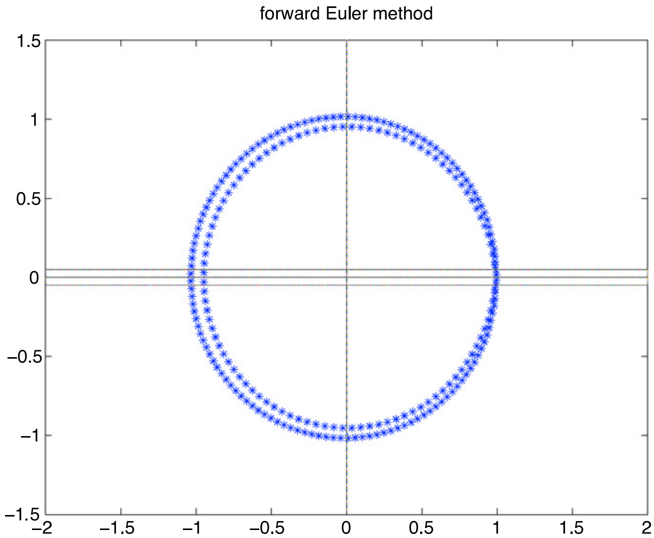

Fig. 3. Characteristic plot for $h = 0.05,\ M = 20$.

Table 2
Cf. Fig. 3: Number of zeros of the polynomial by Rouché's Theorem.

Step length h	Length of rectangle M	Number of zeros N_p
0.05	2	78
0.05	4	38
0.05	10	14
0.05	20	8
0.05	Large	0

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 10\big(5 - (2N_1 + j)h\big)}}{2 \times (-5/2)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$\theta_l = \frac{4 - \sqrt{16 + 10\big(5 - N_4 h + lh\big)}}{2 \times (-5/2)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ'_j and θ'_l are determined by

$$\theta'_j = \frac{9 + \sqrt{81 + 20\big(1 - (2M_1 + j)h\big)}}{2 \times (-5)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$\theta'_l = \frac{9 - \sqrt{81 + 20\big(1 - M_4 h + lh\big)}}{2 \times (-5)}, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the Eq. (30). See Fig. 3 and Table 2.

Example 3.4. Consider the Eq. (1) for the Example 2.4

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta). \tag{31}$$

Here

$$\nu(\theta) = (-\theta - 1)(4\theta + 3), \qquad \eta(\theta) = -8\theta - 8,$$

and

$$r(\theta) = -2\theta^2 - 3\theta + 1, \qquad \tau(\theta) = -\theta + 1.$$

We now find the discrete characteristic polynomial of (31). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -4\theta - 3 = 0$. We get $\theta_r = -\frac{3}{4}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} = 2 > 0,\ r(0) = r_0 = 1 > 0$ and $r(\theta_r) = r_c = \frac{17}{8}$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{2}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{17}{8h},$$

where we choose $m_r = 1$ which guarantees that $N_2 \geq 0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4 = \frac{r_0}{h} = \frac{1}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{17}{8h} - N_4.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -1$. Further it is easy to find that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 2 > 0,\ \tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = 2$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{1}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{2}{h} - M_4.$$

Finally we denote $N = \max\{N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (31)

$$\begin{aligned} P(z) = -z^{N+1} + z^N + h\Bigg[&\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big) \\ &+ \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Bigg]. \end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -2\theta_j^2 - 3\theta_j + 1 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \dots, -N_2,$$

and

$$r(\theta_l) = -2\theta_l^2 - 3\theta_l + 1 = N_4h - lh, \quad l = -N_3, -N_3 + 1, \dots, -1, 0$$

which implies that

$$\theta_j = \frac{3 + \sqrt{9 + 8\big(1 - (2N_1 + j)h\big)}}{2 \times (-2)}, \quad j = -N_1, -N_1 + 1, \dots, -N_2,$$

and

$$\theta_l = \frac{3 - \sqrt{9 + 8\big(1 - N_4h + lh\big)}}{2 \times (-2)}, \quad l = -N_3, -N_3 + 1, \dots, -1, 0.$$

Similarly, θ'_l are determined by

$$\theta'_l = -M_4h + lh + 1, \quad l = -M_3, -M_3 + 1, \dots, -1, 0.$$

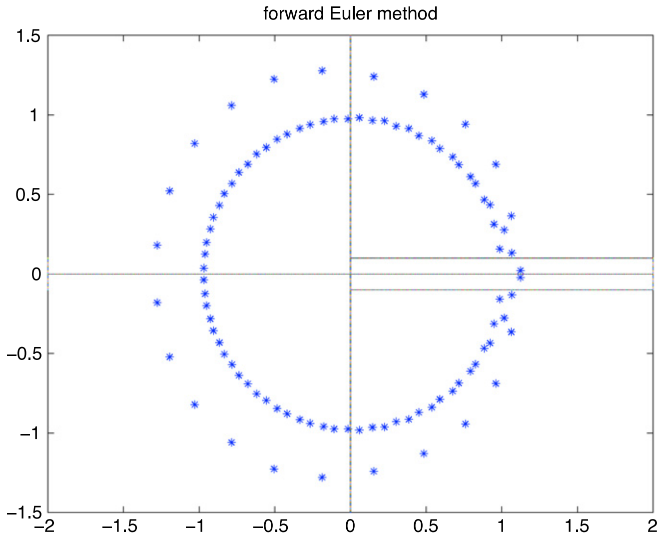

Fig. 4. Characteristic plot for $h = 0.05,\ M = 10$.

Table 3
Cf. Fig. 4: Number of zeros of the polynomial by Rouché's Theorem.

Step length h	Length of rectangle M	Number of zeros N_p
0.05	2	6
0.05	8	2
0.05	10	2
0.05	20	2
0.05	30	2
0.05	Large	0

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the Eq. (31). See Fig. 4 and Table 3.

Example 3.5. Consider the Eq. (1) for the Example 2.5

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta). \tag{32}$$

Here

$$\nu(\theta) = \begin{cases} \theta + 1, & -1 \le \theta < 0, \\ 0, & \theta = 0, \end{cases}$$

and

$$\eta(\theta) = -\theta - 1, \qquad r(\theta) = -\theta^2 + 2, \qquad \tau(\theta) = -\theta + 3.$$

We now find the discrete characteristic polynomial of (32). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -2\theta = 0$. We get $\theta_r = 0$. Further it is easy to see that $r(\theta)$ is increasing on $[-1, \theta_r]$ and $r(-1) = r_{-1} = 1 > 0,\ r(0) = r_0 = 2 > 0$ and $r(\theta_r) = r_c = 2$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{1}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = 2N_1 - \frac{2}{h},$$

where we choose $m_r = 1$ which guarantees that $N_2 \ge 0$.

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -1$. Further it is easy to see that $\tau(\theta)$ is decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = 4 > 0,\ \tau(0) = \tau_0 = 3 > 0$ and $\tau(\theta_\tau) = \tau_c = 4$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{3}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{h}.$$

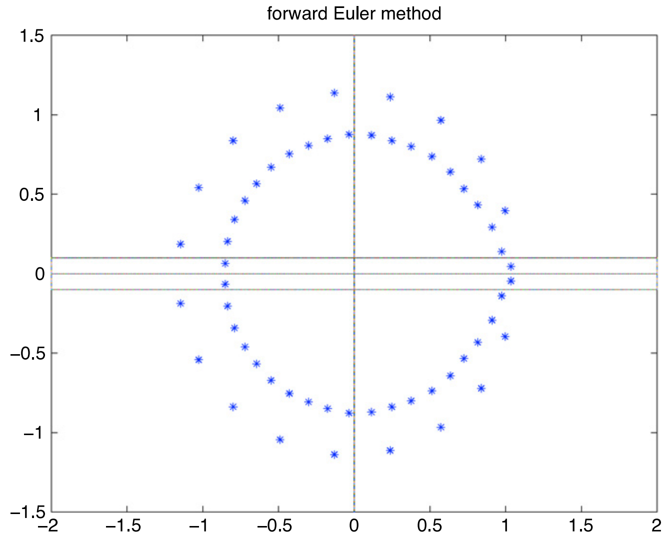

Fig. 5. Characteristic plot for $h = 0.01,\ M = 10$.

Table 4
Cf. Fig. 5: Number of zeros of polynomial by Rouché's Theorem.

Step length h	Length of rectangle M	Number of zeros N_p
0.01	2	10
0.01	4	4
0.01	8	2
0.01	10	2
0.01	20	2
0.01	Large	0

Finally we denote $\max\{N = N_1 + (N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (32)

$$P(z) = -z^{N+1} + z^N + h\left[\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\right].$$

Here θ_j are determined by

$$r(\theta_j) = -\theta_j^2 + 2 = (2N_1 + j)h, \quad j = -N_1, -N_1 + 1, \dots, -N_2,$$

which implies that

$$\theta_j = -\sqrt{2 - (2N_1 + j)h}, \quad j = -N_1, -N_1 + 1, \dots, -N_2.$$

Similarly, θ'_l are determined by

$$\tau(\theta'_l) = -\theta'_l - 1 = M_4 h - lh,$$

which implies that

$$\theta'_l = -M_4 h + lh - 1, \quad l = -M_3, -M_3 + 1, \dots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the Eq. (32). See Fig. 5 and Table 4.

Example 3.6. Consider the Eq. (1) for the Example 2.6

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,\mathrm{d}\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,\mathrm{d}\eta(\theta). \tag{33}$$

Here

$$\nu(\theta) = -(5\theta + 1)(\theta + 1), \qquad \eta(\theta) = -(6\theta + 1)(\theta + 1),$$

and

$$r(\theta) = -10\theta^2 - 4\theta + 10, \qquad \tau(\theta) = -3\theta^2 - \theta + 1.$$

We now find the discrete characteristic polynomial of (33). We first find the critical point θ_r of $r(\theta)$ on $[-1, 0]$. Let $r'(\theta) = -20\theta - 4 = 0$. We get $\theta_r = -\frac{1}{5}$. Further it is easy to find that $r(\theta)$ is increasing on $[-1, \theta_r]$ and decreasing on $[\theta_r, 0]$ and $r(-1) = r_{-1} = 4 > 0,\ r(0) = r_0 = 10 > 0$ and $r(\theta_r) = r_c = 10.4$.

The nonnegative integers $N_1, N_2,\ N_1 > N_2$ can be determined by

$$N_1 = \frac{r_{-1}}{h} = \frac{4}{h}, \qquad N_2 = \frac{(m_r + 1)N_1 - \frac{r_c}{h}}{m_r} = \frac{3N_1 - \frac{10.4}{h}}{2},$$

where we choose $m_r = 2$ which guarantees that $N_2 \geq 0$.

The nonnegative integers N_3 and N_4 can be determined by

$$N_4 = \frac{r_0}{h} = \frac{10}{h}, \qquad N_3 = \frac{r_c}{h} - N_4 = \frac{10.4}{h} - N_4.$$

Next we will find the critical point θ_τ of $\tau(\theta)$ on $[-1, 0]$. We get $\theta_\tau = -\frac{1}{6}$. Further it is easy to find that $\tau(\theta)$ is increasing on $[-1, \theta_\tau]$ and decreasing on $[\theta_\tau, 0]$ and $\tau(-1) = \tau_{-1} = -1 < 0,\ \tau(0) = \tau_0 = 1 > 0$ and $\tau(\theta_\tau) = \tau_c = \frac{13}{12}$.

Note that here $\tau(-1) = \tau_{-1} = -1 < 0$. Let us discretize the integral $\int_{-1}^{\theta_\tau} x(t + \tau(\theta))\, \mathrm{d}\eta(\theta)$. We need to find some nonnegative integers $M_1, M_2,\ M_1 > M_2$ such that $-1 = \theta_{-M_1} < \theta_{-M_1+1} < \cdots < \theta_{-M_2} = \theta_\tau$ is a partition of $[-1, \theta_\tau]$ and

$$\begin{aligned}
&\tau(\theta_{-M_1}) = \tau(-1) = \tau_{-1} = -M_1 h,\\
&\tau(\theta_j) = -M_1 h + m_\tau(M_1 + j)h, \quad j = -M_1 + 1, -M_1 + 2, \ldots, -M_2 - 1,\\
&\tau(\theta_{-M_2}) = \tau(\theta_\tau) = \tau_c = -M_1 h + m_\tau(M_1 - M_2)h.
\end{aligned}$$

Here m_τ is some positive integer which guarantees that $M_2 \geq 0$. In fact, we can determine $M_1, M_2,\ M_1 > M_2$ by the following:

$$M_1 = -\frac{\tau_{-1}}{h} = \frac{1}{h},$$

and, with $m_\tau = 3$,

$$-M_1 h + m_\tau(M_1 - M_2)h = \frac{13}{12},$$

which implies that $M_2 = \frac{2M_1 - \frac{13}{12h}}{3} > 0$. We remark that the bigger m_r is, the faster $\tau(\theta_j),\ j = -M_1 + 1, -M_1 + 2, \ldots, -M_2$ increase. In order to guarantee M_2 is nonnegative, we need to choose $m_r \geq 3$.

The nonnegative integers M_3 and M_4 can be determined by

$$M_4 = \frac{\tau_0}{h} = \frac{21}{h}, \qquad M_3 = \frac{\tau_c}{h} - M_4 = \frac{1}{12h}.$$

Finally we denote $N = \max\{N_1 + m_r(N_1 - N_2 - 1), N_3 + N_4\}$. Then we obtain the following discrete characteristic equation of (33)

$$\begin{aligned}
P(z) = -z^{N+1} + z^N + h\Bigg[&\sum_{j=-N_1}^{-N_2-1} z^{N-[N_1+2m_r(N_1+j)]}\big(\nu(\theta_{j+1}) - \nu(\theta_j)\big) + \sum_{l=-N_3}^{-1} z^{N-[N_3+N_4-(N_3+l)]}\big(\nu(\theta_{l+1}) - \nu(\theta_l)\big)\\
&+ \sum_{j=-M_1}^{-M_2-1} z^{N+[M_1+m_\tau(M_1+j)]}\big(\eta(\theta'_{j+1}) - \eta(\theta'_j)\big) + \sum_{l=-M_3}^{-1} z^{N+[M_3+M_4-(M_3+l)]}\big(\eta(\theta'_{l+1}) - \eta(\theta'_l)\big)\Bigg].
\end{aligned}$$

Here θ_j and θ_l are determined by

$$r(\theta_j) = -10\theta_j^2 - 4\theta_j + 10 = N_1 h + m_r(N_1 + j)h, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

and

$$r(\theta_l) = -10\theta_l^2 - 4\theta_l + 10 = N_4 h - lh, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0$$

which implies that

$$\theta_j = \frac{4 + \sqrt{16 + 40\big(10 - \big(N_1 h + m_r(N_1 + j)h\big)\big)}}{2 \times (-10)}, \quad j = -N_1, -N_1 + 1, \ldots, -N_2,$$

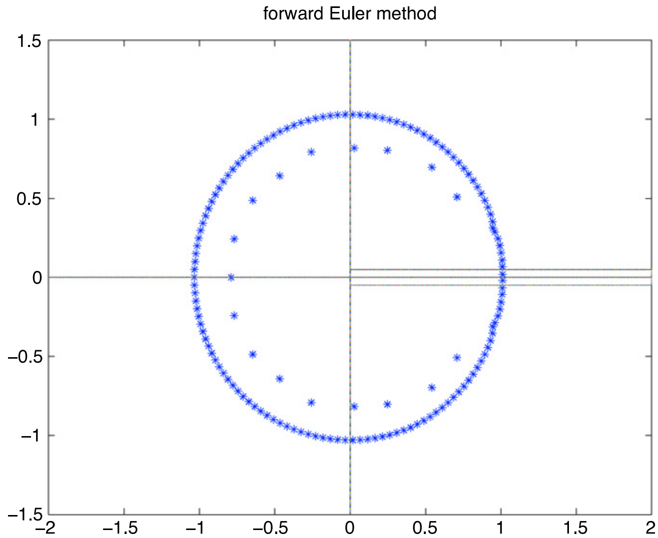

Fig. 6. Characteristic plot for $h = 0.05$, $M = 20$.

Table 5
Cf. Fig. 6: Number of zeros of the polynomial by Rouché's Theorem.

Step length h	Length of rectangle M	Number of zeros N_p
0.05	2	24
0.05	4	12
0.05	8	6
0.05	10	4
0.05	20	2
0.05	30	2
0.05	Large	0

and

$$\theta_l = \frac{4 - \sqrt{16 + 40\big(10 - N_4 h + lh\big)}}{2 \times (-10)}, \quad l = -N_3, -N_3 + 1, \ldots, -1, 0.$$

Similarly, θ_j' and θ_l' are determined by

$$\theta_j' = \frac{1 + \sqrt{1 + 12\big(1 + M_1 h - m_\tau (M_1 + j)h\big)}}{2 \times (-3)}, \quad j = -M_1, -M_1 + 1, \ldots, -M_2,$$

and

$$\theta_l' = \frac{1 - \sqrt{1 + 12\big(1 - M_4 h + lh\big)}}{2 \times (-3)}, \quad l = -M_3, -M_3 + 1, \ldots, -1, 0.$$

Applying Rouché's Theorem, we find that $P(z)$ has no positive real roots which satisfies the conditions for oscillation of the discrete equation. Hence the numerical results are consistent with the theoretical results about the oscillation of the Eq. (33). See Fig. 6 and Table 5.

4. Conclusions

As we have seen, the numerical approach introduced here does provide a reliable method for determining whether or not linear mixed functional differential equations are oscillatory. Based on the experiments we have tried, the technique works also for nonlinear problems, but there is a need for further analytical results in this case.

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